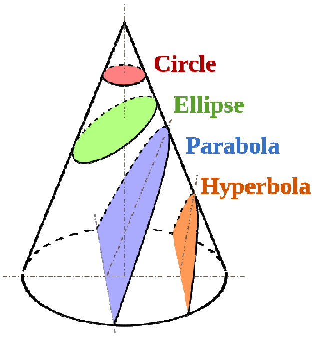
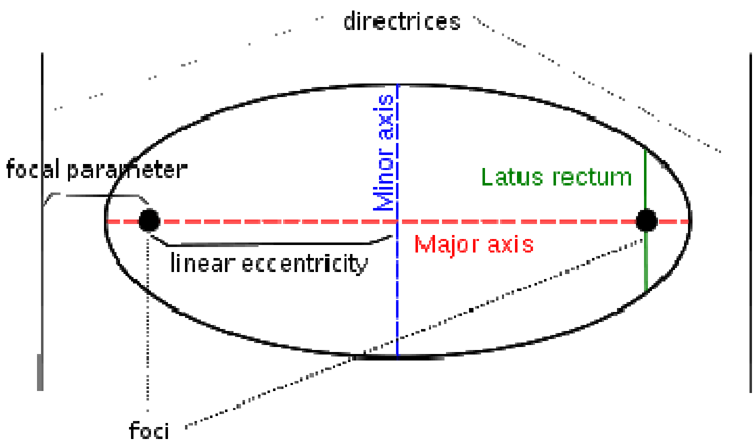
CONIC SECTION

A **conic section** (or just **conic**) is a curve obtained as the intersection of a cone (more precisely, a right circular conical surface) with a plane. In analytic geometry, a conic may be defined as a plane algebraic curve of degree 2. There are a number of other geometric definitions possible. One of the most useful, in that it involves only the plane, is that a conic consists of those points whose distances to some point, called a *focus*, and some line, called a *directrix*, are in a fixed ratio, called the *eccentricity*.
The three types of conic section are the hyperbola, the parabola, and the ellipse. The circle is a special case of the ellipse, and is of sufficient interest in its own right that it is sometimes called the fourth type of conic section. The circle and the ellipse arise when the intersection of cone and plane is a closed curve. The circle is obtained when the cutting plane is parallel to the plane of the generating circle of the cone – for a right cone as in the picture at the top of the page this means that the cutting plane is perpendicular to the symmetry axis of the cone. If the cutting plane is parallel to exactly one generating line of the cone, then the conic is unbounded and is called a parabola. In the remaining case, the figure is a hyperbola. In this case, the plane will intersect *both* halves (*nappes*) of the cone, producing two separate unbounded curves. The type of a conic corresponds to its eccentricity, those with eccentricity less than 1 being ellipses, those with eccentricity equals 1 being parabolas, and those with eccentricity greater than 1 being hyperbolas. In the focus-directrix definition of a conic the circle is a limiting case with eccentricity 0.

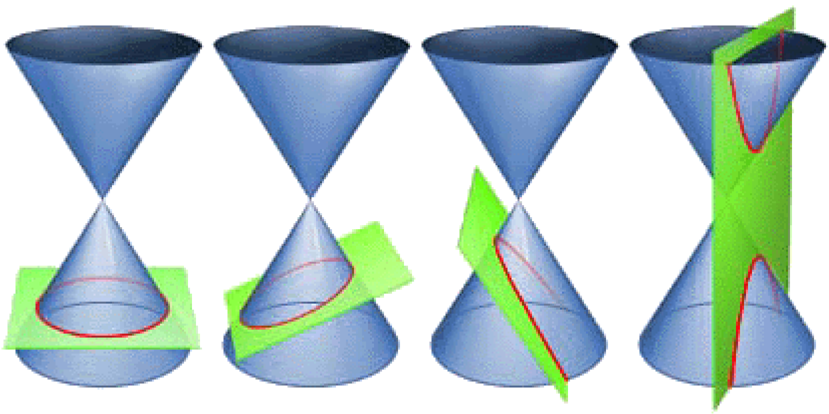


Various parameters are associated with a conic section, as shown in the following table. (For the ellipse, the table gives the case of *a*>*b*, for which the major axis is horizontal; for the reverse case, interchange the symbols *a* and *b*. For the hyperbola the east-west opening case is given. In all cases, *a* and *b* are positive.)

conic section	equation	eccentricity (*e*)	linear eccentricity (*c*)	semi-latus rectum (*ℓ*)	focal parameter (*p*)
circle	$x^2 + y^2 = a^2$	0	0	a	∞
ellipse	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$\sqrt{1 - \frac{b^2}{a^2}}$	$\sqrt{a^2 - b^2}$	$\frac{b^2}{a}$	$\frac{b^2}{\sqrt{a^2 - b^2}}$
parabola	$y^2 = 4ax$	1	a	$2a$	$2a$
hyperbola	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$\sqrt{1 + \frac{b^2}{a^2}}$	$\sqrt{a^2 + b^2}$	$\frac{b^2}{a}$	$\frac{b^2}{\sqrt{a^2 + b^2}}$



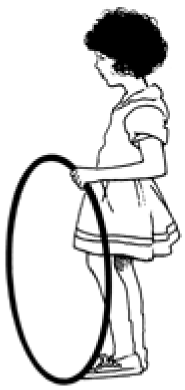
Conics were first studied by one of Plato's pupils. No important scientific applications were found for them until the 17th century, when Kepler discovered that planets move in ellipses and Galileo proved that projectiles travel in parabolas. Appolonious of Perga, a 3rd century B.C. Greek geometer, wrote the greatest treatise on the curves. His work "Conics" was the first to show how all three curves, along with the circle, could be obtained by slicing the same right circular cone at continuously varying angles.



OCCURRENCE OF THE CONICS

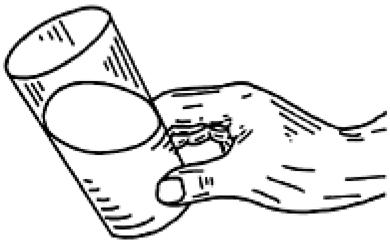
<u>THE ELLIPSE</u>

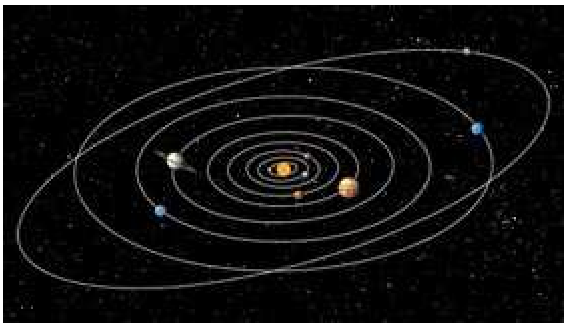
Though not so simple as the circle, the ellipse is nevertheless the curve most often "seen" in everyday life. The reason is that every circle, viewed obliquely, appears elliptical.



Any cylinder sliced on an angle will reveal an ellipse in cross-section (as seen in the Tycho Brahe Planetarium in Copenhagen).

Tilt a glass of water and the surface of the liquid acquires an elliptical outline. Salami is often cut obliquely to obtain elliptical slices which are larger.

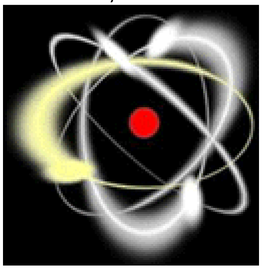




The early Greek astronomers thought that the planets moved in circular orbits about an unmoving earth, since the circle is the simplest mathematical curve. In the 17th century, Johannes Kepler eventually discovered that each planet travels around the sun in an elliptical orbit with the sun at one of its foci.

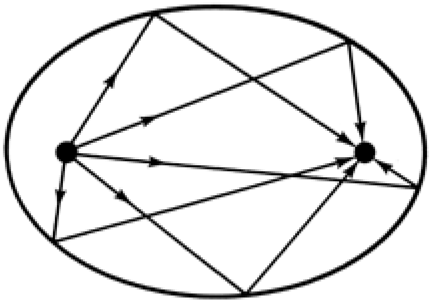
The orbits of the moon and of artificial satellites of the earth are also elliptical as are the paths of comets in permanent orbit around the sun.

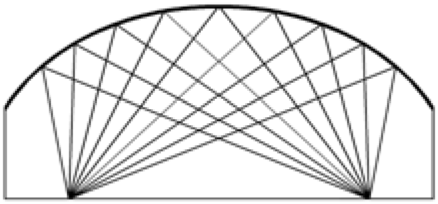
Halley's Comet takes about 76 years to travel abound our sun. Edmund Halley saw the comet in 1682 and correctly predicted its return in 1759. Although he did not live long enough to see his prediction come true, the comet is named in his honour.



On a far smaller scale, the electrons of an atom move in an approximately elliptical orbit with the nucleus at one focus.

The ellipse has an important property that is used in the reflection of light and sound waves. Any light or signal that starts at one focus will be reflected to the other focus. This principle is used in *lithotripsy*, a medical procedure for treating kidney stones. The patient is placed in a elliptical tank of water, with the kidney stone at one focus. High-energy shock waves generated at the other focus are concentrated on the stone, pulverizing it.





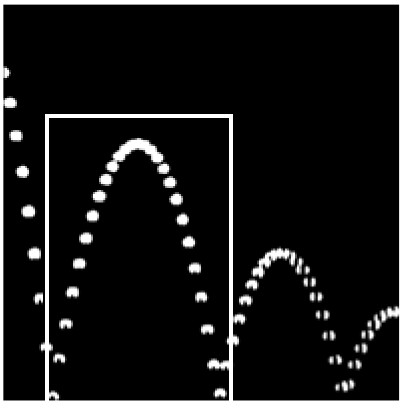
The principle is also used in the construction of "whispering galleries" such as in St. Paul's Cathedral in London. If a person whispers near one focus, he can be heard at the other focus, although he cannot be heard at many places in between.

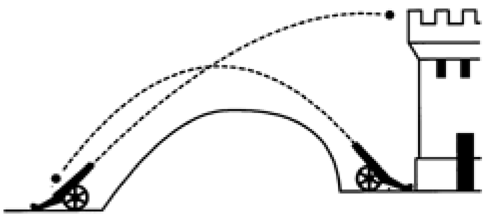
Statuary Hall in the U.S. Capital building is elliptic. It was in this room that John Quincy Adams, while a member of the House of Representatives, discovered this acoustical phenomenon. He situated his desk at a focal point of the elliptical ceiling, easily eavesdropping on the private conversations of other House members located near the other focal point.

The ability of the ellipse to rebound an object starting from one focus to the other focus can be demonstrated with an elliptical billiard table. When a ball is placed at one focus and is thrust with a cue stick, it will rebound to the other focus. If the billiard table is live enough, the ball will continue passing through each focus and rebound to the other.

THE PARABOLA

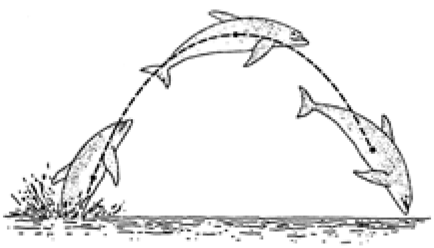
One of nature's best known approximations to parabolas is the path taken by a body projected upward and obliquely to the pull of gravity, as in the parabolic trajectory of a golf ball. The friction of air and the pull of gravity will change slightly the projectile's path from that of a true parabola, but in many cases the error is insignificant.

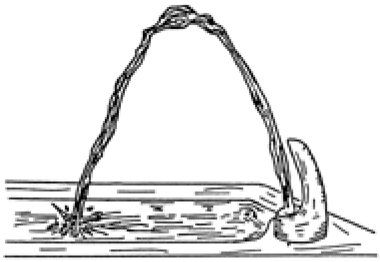




This discovery by Galileo in the 17th century made it possible for cannoneers to work out the kind of path a cannonball would travel if it were hurtled through the air at a specific angle.

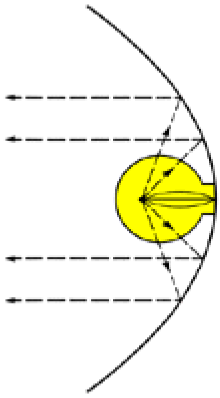
When a baseball is hit into the air, it follows a parabolic path; the center of gravity of a leaping porpoise describes a parabola.

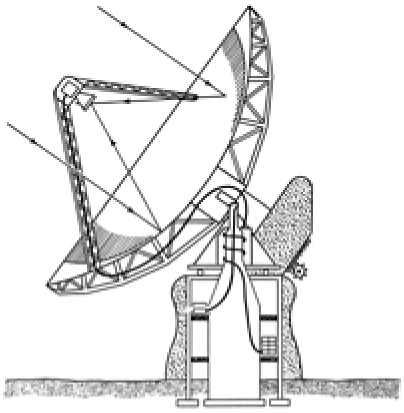




The easiest way to visualize the path of a projectile is to observe a waterspout. Each molecule of water follows the same path and, therefore, reveals a picture of the curve. The fountains of the Bellagio Hotel in Las Vegas comprise a parabolic chorus line.

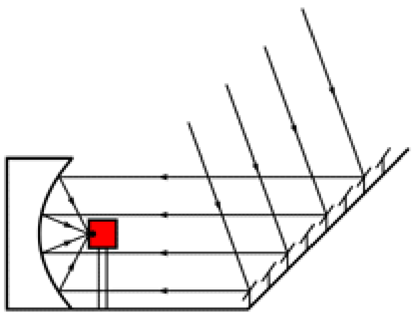
Parabolas exhibit unusual and useful reflective properties. If a light is placed at the focus of a parabolic mirror (a curved surface formed by rotating a parabola about its axis), the light will be reflected in rays parallel to said axis. In this way a straight beam of light is formed. It is for this reason that parabolic surfaces are used for headlamp reflectors. The bulb is placed at the focus for the high beam and a little above the focus for the low beam.

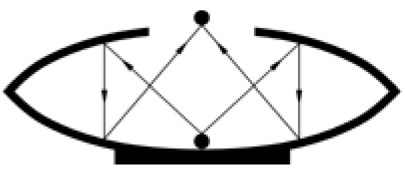




The opposite principle is used in the giant mirrors in reflecting telescopes and in antennas used to collect light and radio waves from outer space: the beam comes toward the parabolic surface and is brought into focus at the focal point. The instrument with the largest single-piece parabolic mirror is the Subaru telescope at the summit of Mauna Kea in Hawaii (effective diameter: 8.2 m).

Heat waves, as well as light and sound waves, are reflected to the focal point of a parabolic surface. If a parabolic reflector is turned toward the sun, flammable material placed at the focus may ignite. (The word "focus" comes from the Latin and means *fireplace*.) A solar furnace produces heat by focusing sunlight by means of a parabolic mirror arrangement. Light is sent to it by set of moveable mirrors computerized to follow the sun during the day. Solar cooking involves a similar use of a parabolic mirror.



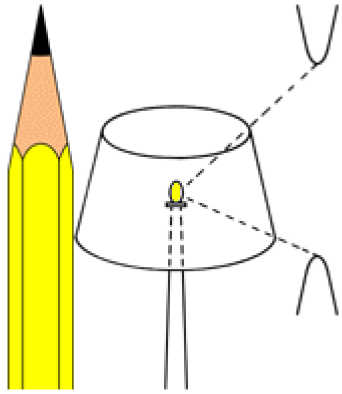


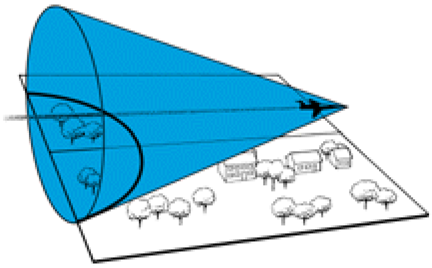
Two types of images exist in nature: real and virtual. In a real image, the light rays actually come from the image. In a virtual image, they appear to come from the reflected image - but do not. For example, the virtual image of an object in a flat mirror is "inside" the mirror, but light rays do not emanate from there. Real images can form "outside" the system, where emerging light rays cross and are caught - as in a *Mirage*, an arrangement of two concave parabolic mirrors.

Mirage's 3-D illusions are similar to the holograms formed by lasers.

THE HYPERBOLA

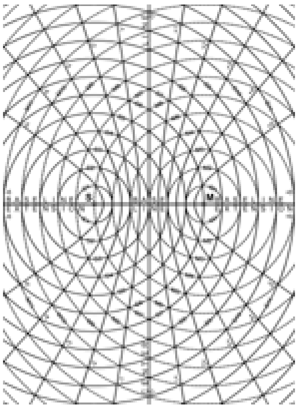
If a right circular cone is intersected by a plane parallel to its axis, part of a hyperbola is formed. Such an intersection can occur in physical situations as simple as sharpening a pencil that has a polygonal cross section or in the patterns formed on a wall by a lamp shade.





A sonic boom shock wave has the shape of a cone, and it intersects the ground in part of a hyperbola. It hits every point on this curve at the same time, so that people in different places along the curve on the ground hear it at the same time. Because the airplane is moving forward, the hyperbolic curve moves forward and eventually the boom can be heard by everyone in its path.

A hyperbola revolving around its axis forms a surface called a hyperboloid. The cooling tower of a steam power plant has the shape of a hyperboloid, as does the architecture of the James S. McDonnell Planetarium of the St. Louis Science Center.



All three conic sections can be characterized by moiré patterns. If the center of each of two sets of concentric circles is the source of a radio signal, the synchronized signals would intersect one another in associated hyperbolas. This principle forms the basis of a hyperbolic radio navigation system known as *Loran* (**L**ong **R**ange **N**avigation).