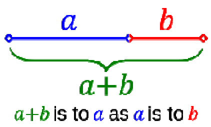
GOLDEN RATIO

In mathematics, two quantities are in the golden ratio if the ratio between the sum of those quantities and the larger one is the same as the ratio between the larger one and the smaller.
I want to say:



$$\frac{a+b}{a}=\frac{a}{b}$$

In this way, the golden section is a line segment sectioned into two according to the golden ratio: the total length is to the longer segment x as x is to the shorter segment.

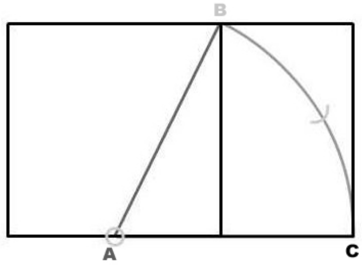
Some of the greatest mathematical minds of all ages such as Pythagoras and Euclid in ancient Greece, Leonardo of Pisa (known as Fibonacci), astronomers such as Kepler or even present day scientific figures such as oxford physicist Roger Penroe, have spent endless hours over this simple ratio and its properties. Nevertheless the fascination with the golden ratio isn't confined just to mathematicians but also to biologists, artists, musicians, historians, architects, psychologists and even mystics.
The golden ratio is usually denoted by the Greek letter Φ.
What's exactly Φ?

Let's go to this expression: $\frac{x+y}{x}=\frac{x}{y}$. We can call the ratio $\frac{x}{y}$= α. Now we can express $\frac{x}{x}+\frac{y}{x}=\frac{x}{y}$ which is equivalent to $1+\frac{1}{\alpha}=\alpha \leftrightarrow \alpha^2-\alpha-1=0 \leftrightarrow$

$$\alpha = \frac{1+\sqrt{1-4.1.(-1)}}{2.1}=\frac{1+\sqrt{5}}{2}= \begin{cases} \frac{1+\sqrt{5}}{2}=\Phi \\ \frac{1-\sqrt{5}}{2} \end{cases}$$

This second degree equation has as its unique positive solution the irrational number $\Phi=\frac{1+\sqrt{5}}{2}$=1'618033987…

Other names frequently used for the golden ratio are: golden section, golden mean, golden number, extreme and mean ratio, divine proportion, divine section…
At least since the Renaissance, many artists and architects have used the golden ratio especially in the form of the **golden rectangle**, in which the ratio of the longer side to the shorter one is the golden ratio, believing this proportion to be aesthetically pleasing.
To construct a golden rectangle we have to start by drawing a square of x units by x units, then drawing a line from the midpoint of one side of the square to one of the corners on the opposite side of the square:



Using that line as the radius to draw an arc, this arc defines the long dimension of the rectangle. The completed rectangle is in the proportion of the golden ratio. To proof that, all we have to do is to notice that the triangle ADB is also a right-angled triangle. Therefore, via Pythagoras theorem, the square of the hypotenuse is equal the sum of the squares of the other two sides:

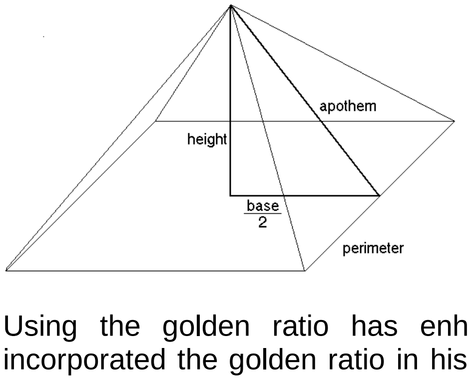
$$h^2 = x^2 + \left(\frac{x}{2}\right) \leftrightarrow h^2 = \frac{5}{4}x^2 \leftrightarrow h = \frac{\sqrt{5}}{2}x.$$

Calculating the ratio: $\frac{\frac{x}{2}+\frac{\sqrt{5}x}{2}}{x} = \frac{1+\sqrt{5}}{2} = \Phi$, so we can see that in fact this rectangle is a golden rectangle.

If you ever need an easily accessible example of a golden rectangle, all you have to do is to pull out a credit card or an ID.

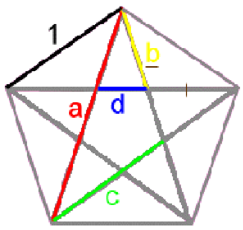
Phidias (500 BC-432 BC), a Greek sculptor and mathematician, applied Φ to design the sculptures that appear in the Parthenon. The Parthenon’s façade is circumscribed by golden rectangles as well.

Other examples are: the Great Mosque of Kairouan, the Notre Dame cathedral in Paris or several buildings and even some Cromlechs. Also the ancient Egyptians recorded the knowledge of Φ thousands years ago in their architecture. The great Pyramid of Cheops is a good example. In the Pyramid of Cheops the sloping faces give a slope height of Φ times half the base. I mean $\Phi = \frac{apothem}{\frac{base}{2}}$



Using the golden ratio has enhanced many works of art. Leonardo da Vinci incorporated the golden ratio in his own paintings. His Mona Lisa is a clear example. Salvador Dalí used the golden ratio in his masterpiece the Last Supper. Many paintings often use a rectangular canvas with a golden ratio because it has a better look than another different canvas such as a squared.

The golden ratio plays an important role in rectangular pentagons and pentagrams. Each intersection of diagonals is according to the golden ratio.



In fact, if we consider two diagonals (AC and BD) in this rectangular pentagon, we realise that the triangles OBC and OAD are similar because their angles are equal. Then $\frac{OA}{OC} = \frac{AD}{BC}$, as well as AD=AC and BC=ED=OA, which yields $\frac{OA}{OC} = \frac{AD}{OA} \leftrightarrow \frac{AC}{OA} = \frac{OA}{OC}$=Φ. So, the point O divides AC in extreme and mean ratio.

Another important relationship is Φ and Fibonacci sequence. The **Fibonacci sequence** is a recurrent sequence whose expression is: $a_{n+2} = a_{n+1} + a_n$, where the sum of the two adjacent numbers in the sequence forms the next higher number:

1,1,2,3,5,8,13,21,34,55,89,144,…

This sequence is given on solving the numbers of rabbits that will be after one year, starting from a couple of rabbits, if the rabbits and each new pair of rabbits gave birth to a new pair of rabbits every month.

The golden ratio is the limit of the ratios of successive terms of the Fibonacci sequence:

$\lim \frac{a}{a}$=Φ.

In music the scale is based on Fibonacci numbers. For instance, the piano keyboard has 8 white keys and 5 black keys making 13 keys in total.

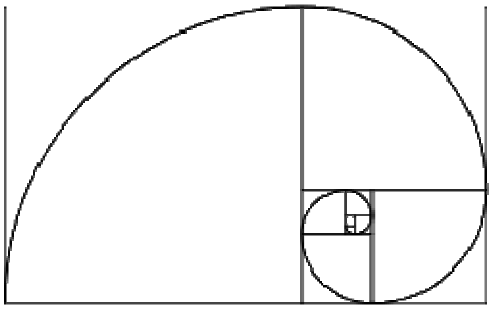
Other alternate forms of Φ are:

$$\varphi = [1; 1, 1, 1, \dots] = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}}$$

.

$$\varphi = \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \cdots}}}}.$$

Any golden rectangle can be divided into a square and a smaller golden rectangle. This process can be continued to infinity. So we can construct this sort of spiral called **Fibonacci spiral** which is a type of logarithmic spiral.

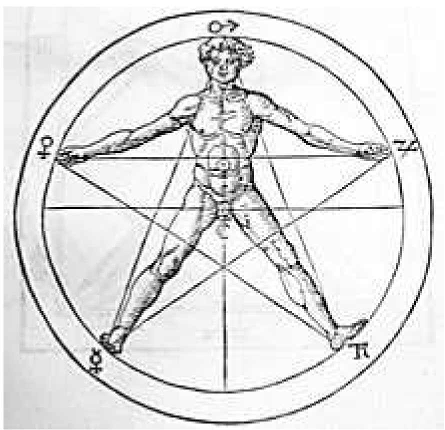


The cochlea of the inner ear is shaped in a Fibonacci spiral, which explains why it sounds so good to us. The tail of a comet curves away from the sun in a Fibonacci spiral. The spider spins webs in a golden spiral. Pine cones, sea horses, snail shells (such as Nautilus), mollusk shells, ocean waves, animal horns, fern and the arrangement of sun flowers and daisy seeds all form golden spirals. Hurricane clouds, galaxy swirls take the form of logarithmic spirals as well.
As we can see Φ is found in many forms in nature. We can also find the golden ratio in the arrangement of branches along the stems of plants and of veins of leaves (Philotaxia). Experimentally we know that the angle of turning from a leave to next one is ≈ 137º30'3''. Mathematically, $\frac{360}{x} = \frac{x}{y}$, $\frac{x}{y} = \Phi$. These expressions yield a system of two equations:

$x = \Phi y$

$360y = x^2 \quad \leftrightarrow 360y = (\Phi y)^2 \leftrightarrow y = \frac{360}{\quad}$ ≈ 137º30'33'', as we expected.

The golden ratio appears as well in the skeleton of animals, in the branches of their veins and nerves and of course, this divine number is closely connected with the beauty in the human body: the navel divides the human body in extreme and mean ratio.



To end up, we can consider the following Galileo's phrase:

"The book of Nature is written in mathematical language".