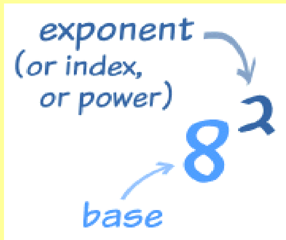
UNIT 6 VOCABULARY: POWERS AND ROOTS

1.1. Powers with natural base

An exponent is a short way of writing the same number multiplied by itself many times. Exponents can also be called **indices** (singular index).

In general, for any real number a and natural number n, we can write a multiplied by itself n times as a^n .

$$a^n = \underbrace{a \times a \times \ldots \times a}_{n}$$



READING POWERS!

A power can be read in many ways in English. For example, 6^5 can be read as:

- The fifth power of six.
- Six powered to five.
- **The most common one: six to the power of five.**

There are two especial cases: **powers of two and three**.

- 3^2 is read **three squared**.
- 5^3 is read **five cubed**.

Exercises.

1. Calculate mentally and write in words the following powers:

   a) 4^3 **c)** 11^2 **e)** 5^3
   b) 5^4 **d)** 2^5 **f)** 10^3

2. Match the following numbers to their squares.

   169 196 25 81 400 10000

   20^2 13^2 9^2 5^2 100^2 14^2

3. Match the following numbers to their cubes.

   125 8000 1 64 1000 27

   20^3 4^3 10^3 3^3 5^3 1^3

4. Fill in the missing numbers:

   a) $5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 = 5^{[\ ]}$ **c)** $10000000 = 10^{[\ ]}$ **e)** $16 = [\ ]^2$
   b) $8 \cdot 8 \cdot 8 \cdot 8 = 8^{[\ ]}$ **d)** $81 = [\ ]^2$ **f)** $16 = 2^{[\ ]}$

1.2. Perfect squares and perfect cubes

The **perfect squares** are the squares of the natural numbers:

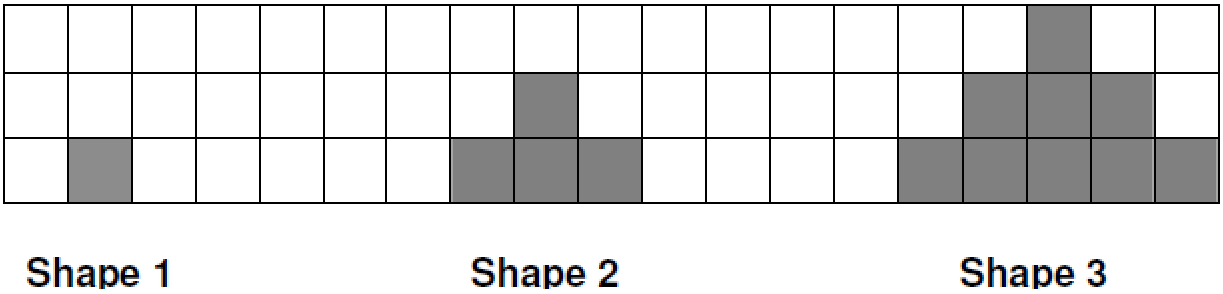
Natural number	1	2	3	4	5	6	7	8	9	10	11	12	13	...
Perfect Squares	**1**	**4**	**9**	**16**	**25**	**36**	**49**	**64**	**81**	**100**	**121**	**144**	**169**	**...**

The perfect cubes are the cubes of the natural numbers:

Natural Number	1	2	3	4	5	6	7	8	9	10	11	12	13	...
Perfect Cubes	1	8	27	64	125	216	343	512	729	1000	1331	1728	2197	...

Exercises.

1. You can build up a pattern using square tiles.



a) Draw the next two shapes in the pattern.

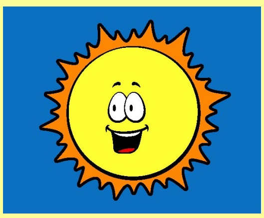
b) Count the numbers of tiles in each shape.
c) How many tiles are there in:
shape 6? shape 9? shape 15?
d) Explain how to know the number of tiles when you know the number of the shape.

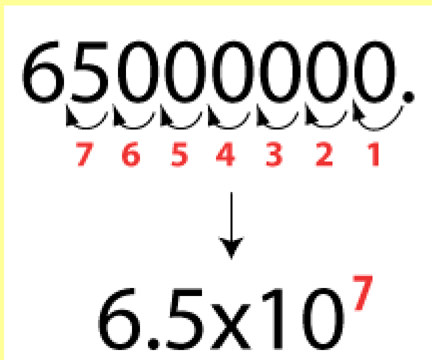
1.3. Standard form

Standard form is a special way of writing numbers that makes it easier to use big and small numbers.

It is very useful in Science. For example, the Sun has a Mass of 1,988,000,000,000,000,000,000,000,000,000 kg

It is much more comfortable to write 1.988×10^{30} kg.





So the number is written in **two parts**:

- **The digits**, with **the decimal point placed after the first digit**, followed by
- **× 10 to a power** that shows how many places to move the decimal point.

Exercises.

1. Express in standard form the following numbers:

   a) 4,000,000,000 **c)** 321,650,000 (round to the million)

   b) A billion **d)** The number of seconds in a year (round appropriately)

2. Write as ordinary numbers:

   a) $3.4 \cdot 10^5$ **c)** $0.05 \cdot 10^2$ **e)** $2.473 \cdot 10^8$

   b) $7.26 \cdot 10^2$ **d)** $7.006 \cdot 10^7$ **f)** $9 \cdot 10^{12}$

1.3. Powers with negative base

The sign of a power is positive, **unless the base is negative and the exponent is an odd number**.

Base	Exponent	Sign of the result	Example
+	Odd or even	+	$2^3=8$ $2^4=16$
-	Even	+	$(-2)^4=(-2)\cdot(-2)\cdot(-2)\cdot(-2)=16$
-	Odd	-	$(-2)^3=(-2)\cdot(-2)\cdot(-2)=-8$

BE CAREFUL WITH BRACKETS!!!!

Brackets are very important in Maths, and you have to be very careful with them. Have a look at the following examples:

With ( )	$(-2)^2=(-2)\cdot(-2)=4$
Without ( )	$-2^2=-2\cdot 2=-4$

With ( )	$(-2)^3=(-2)\cdot(-2)\cdot(-2)=-8$
Without ( )	$-2^3=-2\cdot 2\cdot 2=-8$

Exercises.

1. Calculate mentally:
   - **a)** $(-4)^3$
   - **b)** $(-5)^4$
   - **c)** $(-11)^2$
   - **d)** $(-2)^5$
   - **e)** $(-5)^3$
   - **f)** $(-10)^3$
2. Calculate mentally:
   - **a)** -4^3
   - **b)** -5^4
   - **c)** -11^2
   - **d)** -2^5
   - **e)** -5^3
   - **f)** -10^3

2.1. Laws of exponents

There are several laws we can use to make working with exponential numbers easier.

LAW	EXAMPLE
$x^0=1$	$7^0=1$
$a^1=a$	$5^1=5$
$a^m \cdot a^n=a^{m+n}$	$x^2 \cdot x^3=x^{2+3}=x^5$
$\frac{a^m}{a^n}=a^{m-n}$ or $a^m:a^n=a^{m-n}$	$\frac{2^6}{2^2}=2^{6-2}=2^4$ or $2^6:2^2=2^{6-2}=2^4$
$(a \cdot b)^n=a^n b^n$	$(2 \cdot 3)^4=2^4 \cdot 3^4$
$\left(\frac{a}{b}\right)^n=\frac{a^n}{b^n}$ $(a:b)^n=a^n:b^n$	$\left(\frac{2}{5}\right)^3=\frac{2^3}{5^3}=\frac{8}{125}$ $(2:5)^3=2^3:5^3$

Exercises.

1. Fill in the missing numbers:
   - **a)** $3^3 \cdot 3^4 = 3^{[\,]}$
   - **b)** $7^5 \cdot 7^8 = 7^{[\,]}$
   - **c)** $6^2 \cdot 6^7 = 6^{[\,]}$
   - **d)** $6^5 \cdot 6^{[\,]} = 6^{[\,]}$
   - **e)** $[\,]^3 \cdot [\,]^4 = 2^7$
   - **f)** $2^5 \cdot 2^{[\,]} = 2^6$
   - **g)** $2^7 \cdot [\,]^{[\,]} = 29$
   - **h)** $x^2 \cdot x^3 \cdot x^4 = x^{[\,]}$
2. Fill in the missing numbers:
   - **a)** $7^5 : 7^2 = 7^{[\,]}$
   - **b)** $12^{13} : 12^7 = 12^{[\,]}$
   - **c)** $3^8 : [\,]^3 = 3^5$
   - **d)** $5^{[\,]} : [\,]^2 = 5^7$
   - **e)** $[\,]^{12} : [\,]^9 = 9^{[\,]}$
   - **f)** $x^5 : x^3 : x^2 = x^{[\,]}$

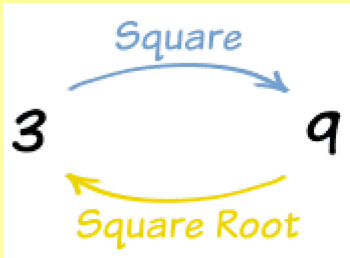
3. Fill in the missing numbers:
   a) $(4^2)^5 = 4^{[\ ]}$ b) $(3^2)^{[\ ]} = 3^8$ c) $(m^{[\ ]})^2 = m^8$ d) $([\ ]^2)^3 = 5^6$ e) $(x^2)^{[\ ]} = [\ ]^6$

4. Fill in the missing numbers:
   a) $3^7 \cdot 8^7 = [\ ]^7$
   b) $[\ ]^2 \cdot [\ ]^2 = 6^2$
   c) $5^2 \cdot [\ ]^2 = 15^2$
   d) $2^2 \cdot [\ ]^2 = 14^{[\ ]}$
   e) $8^5 : 4^5 = [\ ]^5$
   f) $\dfrac{15^5}{5^5} = [\ ]^5$
   g) $\dfrac{16^7}{8^7} = [\ ]^7$
   h) $\dfrac{6^{12}}{3^{12}} = [\ ]^{[\ ]}$

5. Express as a single power:
   a) $7^3 \cdot 7^5$
   b) $4^2 \cdot 4^3 \cdot 4^6 \cdot 4$
   c) $t^2 \cdot t^7 \cdot t^2$
   d) $5^7 : 5^3$
   e) $(2^2 \cdot 2^6) : 2^3$
   f) $(14^2)^4$
   g) $3^7 : (3^2 \cdot 3^3)$
   h) $2^2 \cdot 3^2$
   i) $(12^2 \cdot 12^3)^4$
   j) $6^2 : 3^2$
   k) $[3^8 : (3^2 \cdot 3^3)]^4$
   l) $\dfrac{5^7 \cdot 5^3}{5^4}$

3.1. Square root of a number

A square root of a number is a value that can be **multiplied by itself** to give the original number. A square root is the opposite of a square:



For example, a square root of 9 is 3, because when 3 is multiplied by itself you get 9.

This is the special symbol that means "square root". It is also called ***"radical"*** symbol.

It is very easy to read a square root:

$\sqrt{49} = 7$ is read "the square root of 49 is 7"

In general:

$$\sqrt{a} = b \quad \textbf{because} \quad b^2 = a$$

3.2. Number of square roots

In some cases, there can be more than one root (or less than one!):

$\sqrt{a}$	Number of roots	Example
$a > 0$	Two opposite roots	$\sqrt{81} = \pm 9$ because $9^2 = 81$ and $(-9)^2 = 81$
$a = 0$	One root	$\sqrt{0} = 0$
$a < 0$	No real roots	$\sqrt{-4}$ has no real roots

Exercises.

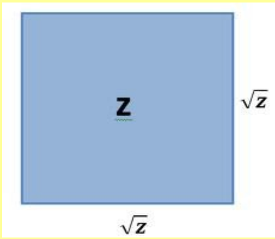
1. Given y^2, find the value of y:
   a) $y^2 = 81$ **b)** $y^2 = 16$ **c)** $y^2 = 100$ **d)** $y^2 = 4$ **e)** $y^2 = 36$

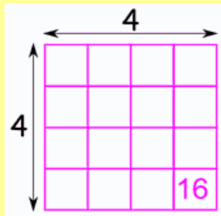
2. Fill in the gaps with the proper word or expression:
   a) The _______ root of 36 is 6 or -6 because _______ and _______ are equal to 36.
   b) The square root of 64 is ____ or ____ because 8^2 and ______ are equal to 64.
   c) The square _____ of _____ is _____ or _____ because $12^2 = 144$ and _______ .
   d) The square root of 10000 is ______ or _____ because _______ and _______ .
   e) The square root of -121 can't be calculated because the _______ is a _______ number.

3. Write the different parts of the expression $\sqrt{169} = 13$

   Radicand ______ **Radical** ______ **Root** ______

If you know the area of a square, **you can use the square root to find the length of the side of the square.** Look:





3.3. Calculating squares roots

It is **easy to work out the square root of a perfect square**, but it is **really hard** to work out other square roots.
$\sqrt{25} = 5$ $\sqrt{169} = 13$ $\sqrt{100} = 10$ $\sqrt{900} = 30$

For example, what is $\sqrt{10}$?

Well, **3 × 3 = 9 and 4 × 4 = 16**, so **we can guess the answer is between 3 and 4.**

$$3 < \sqrt{10} < 4$$

- Let's try 3.5: *3.5 × 3.5 = 12.25*
- Let's try 3.2: *3.2 × 3.2 = 10.24*
- Let's try 3.1: *3.1 × 3.1 = 9.61*

Getting closer to 10, but it will take a long time to get a good answer!

At this point, I get out my calculator and it says:

$\sqrt{10}$ = *3.1622776601683793319988935444327 ...*

But **the digits just go on and on, without any pattern**.

So even the calculator's answer is **only an *approximation* !**

Exercise. Estimate the value of the following square roots:

a) __ < $\sqrt{57}$ < __ **c)** __ < $\sqrt{700}$ < __ **e)** __ < $\sqrt{30}$ < __

b) __ < $\sqrt{250}$ < __ **d)** __ < $\sqrt{1500}$ < __ **f)** __ < $\sqrt{200}$ < __

When we estimate a square root as $3 < \sqrt{14} < 4$, we can also say that **the square root of 14 is 3**, and the difference of 14 and 9 (square of 3), which is **5**, is called the **remainder**.

$$\text{radicand} = \text{root}^2 + \text{remainder}$$

For example, $14 = 3^2 + 5$.

Exercise. Calculate the square roots and the remainders for the numbers of the previous exercise.

3.4. Order of operations

When you have several operations to do, which one do you calculate first?

We work out operations in this order:

BRACKETS

EXPONENTS (Powers, roots, etc)

DIVISION and **M**ULTIPLICATION (working from left to right)

ADDITION and **S**UBTRACTION (working from left to right)

That makes **BEDMAS!**